Flat Beam Producing Adaptor

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Section 1. Flat Beam Producing Condition

Our goal is to use a so-called adaptor, a set of consecutive skew quadrupoles, to manipulate a bunch of particles to get a flat beam, which has a very small emittance in one plane compared to that in the other plane. Using skew quadrupoles in stead of normal quadrupoles is in order to get a flat beam standing in horizontal or vertical plane instead of 45 degree or 135 degree inclined plane. The bunch of particles is born in a finite uniform magnetic field produced by one or several solenoids. Ideally, for a cold bunch (by "cold" I mean zero initial transverse momentum), and space charge effect ignored, the bunch will just drift with initial energy in the magnetic field until it meets the end of the solenoid field. We will stick with this ideal cold bunch until we get some basic idea of the transformation. Actually, If we noticed the form of the matrices of solenoid exit and skew quadrupoles,

the matrix of the exit of solenoid is: $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -K & 0 \\ 0 & 0 & 1 & 0 \\ K & 0 & 0 & 1 \end{pmatrix}, where K = -\dfrac{eBz}{2m\gamma\beta c}$

the matrix of the rotation is: $R_{\frac{\pi}{4}} = \begin{pmatrix} \cos(\pi/4) & 0 & \sin(\pi/4) & 0 \\ 0 & \cos(\pi/4) & 0 & \sin(\pi/4) \\ -\sin(\pi/4) & 0 & \cos(\pi/4) & 0 \\ 0 & -\sin(\pi/4) & 0 & \cos(\pi/4) \end{pmatrix}$

the matrix of the skew quadrupole is: $Q_s = R_{\frac{\pi}{4}}{}^{-1} Q R_{\frac{\pi}{4}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \dfrac{1}{f} & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{1}{f} & 0 & 0 & 1 \end{pmatrix}$

we would understand why we choose these components to achieve the task, the point is that all these matrices are X-Y plane coupled and our task is to transfer emittance from one phase space to the other.

In figure 1, it is a illustration of the solenoid and an adaptor, in reality, the position of each component is fixed, the solenoid field is relatively fixed(the selection of value of the field usually includes some factors other than pursuing flattest bunch), the gradients of the quadrupoles are usually adjustable.

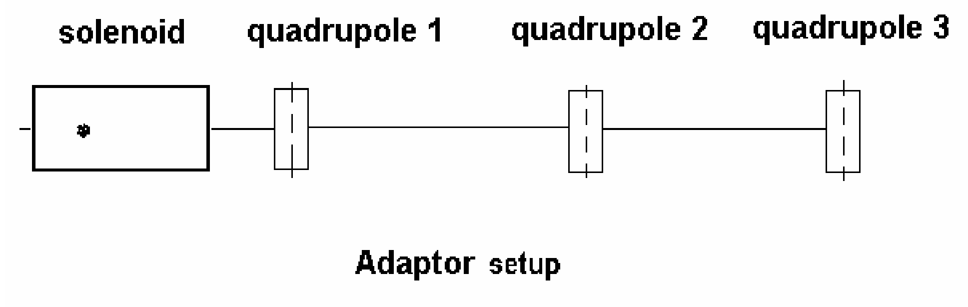

Figure 1. Flat beam producing adaptor setup

Now let's consider how to get flat beam with this setup by matrix approach. At first, we have a cold bunch, the status of any one particles of the bunch can presented as:

$$X_0 = \begin{pmatrix} X_0 \\ 0 \\ Y_0 \\ 0 \end{pmatrix}$$

For simplicity, I am going to use normal quadrupoles in stead of skew quadruples in following analysis, and which will result in a flat beam inclined in a 45 degree plane or 135 degree plane. So, we can express the transfer matrix for the whole adaptor as:

$$T = \begin{pmatrix} M_{11} & M_{12} & 0 & 0 \\ M_{21} & M_{22} & 0 & 0 \\ 0 & 0 & N_{11} & N_{12} \\ 0 & 0 & N_{21} & N_{22} \end{pmatrix}$$

Then, when the bunch arrives the exit of this adaptor, it will be with status:

$$X_1 = \begin{pmatrix} X_1 \\ X_1' \\ Y_1 \\ Y_1' \end{pmatrix} = TCX_0 = \begin{pmatrix} M_{11}X_0 - M_{12}KY_0 \\ M_{21}X_0 - M_{22}KY_0 \\ N_{12}KX_0 + N_{11}Y_0 \\ N_{22}KX_0 + N_{21}Y_0 \end{pmatrix} \qquad (1)$$

To get the flat beam inclined at 45 degree we are pursuing, we need $\begin{cases} X_1 = Y_1 \\ X_1' = Y_1' \end{cases}$. Since this transformation is valid for particle with any position within the bunch, from (1), we know that means:

$$\begin{cases} M_{11} = N_{12}K \\ -M_{12}K = N_{11} \\ M_{21} = N_{22}K \\ -M_{22}K = N_{21} \end{cases} \tag{2}$$

Only three equations in (2) are independent, since the determinant of the transfer matrix is 1. And we will use the first three in following section's calculations. Now Let's express (2) in matrix way, we can find a simple form for this plat beam producing condition,

$$\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{K} \\ -K & 0 \end{pmatrix}$$

$$or: N = MF \tag{3}$$

$$or: M = -NF$$

where N and M are transfer matrixes for each plane of the adaptor and

$$F = \begin{pmatrix} 0 & \frac{1}{K} \\ -K & 0 \end{pmatrix}, F^{-1} = -F$$

For a flat beam inclined at 135 degree, by similar procedures, we can get

$$N = -MF \Leftrightarrow M = NF$$

which is exactly same as that in FERMILAB-TM-2114.

In this paper, an equivalent condition is also given in the form of Courant-Snyder parameters. For a transfer line, the matrix can be represented as;

$$M = \begin{pmatrix} \sqrt{\frac{\beta_{2x}}{\beta_1}}(\cos\mu_x + \alpha_1 \sin\mu_x) & \sqrt{\beta_1\beta_{2x}}\sin\mu_x \\ -\frac{1+\alpha_1\alpha_{2x}}{\sqrt{\beta_1\beta_{2x}}}\sin\mu_x + \frac{\alpha_1 - \alpha_{2x}}{\sqrt{\beta_1\beta_{2x}}}\cos\mu_x & \sqrt{\frac{\beta_1}{\beta_{2x}}}(\cos\mu_x - \alpha_{2x}\sin\mu_x) \end{pmatrix}$$

$$N = \begin{pmatrix} \sqrt{\frac{\beta_{2y}}{\beta_1}}(\cos\mu_y + \alpha_1 \sin\mu_y) & \sqrt{\beta_1\beta_{2y}}\sin\mu_y \\ -\frac{1+\alpha_1\alpha_{2y}}{\sqrt{\beta_1\beta_{2y}}}\sin\mu_y + \frac{\alpha_1 - \alpha_{2y}}{\sqrt{\beta_1\beta_{2y}}}\cos\mu_y & \sqrt{\frac{\beta_1}{\beta_{2y}}}(\cos\mu_y - \alpha_{2y}\sin\mu_y) \end{pmatrix}$$

the subscript 1 and 2 indicate the starting point and ending point of the transfer line. μ is the phase advance. At the starting point of the adaptor, for the cold bunch getting out of the solenoid:

$$X = CX_0 = \begin{pmatrix} X_0 \\ -KY_0 \\ Y_0 \\ KY_0 \end{pmatrix} \tag{4}$$

We can consider that, at initial status, all particles within the bunch are located on circles with different radius, or,

$$X_0^{\,2} + Y_0^{\,2} = R^2$$

from (4), we can derive easily the status of the bunch at the entrance of the adaptor:

$$\begin{cases} KX^2 + \dfrac{1}{K} X'^2 = KR_0^2 \\ KY^2 + \dfrac{1}{K} Y'^2 = KR_0^2 \end{cases}$$

Compare to the beam ellipse equation in trace space:

$$\begin{cases} \gamma_x X^2 + 2\alpha_x XX' + \beta_x X'^2 = \varepsilon \\ \gamma_y Y^2 + 2\alpha_y YY' + \beta_y Y'^2 = \varepsilon \end{cases}$$

It is easily to find that:

$$\begin{cases} \alpha_1 = 0 \\ \beta_1 = \dfrac{1}{K} \\ \varepsilon = KR_0^2 \end{cases} \tag{5}$$

Then the matrix of the adaptor in Courant-Snyder form is:

$$M = \begin{pmatrix} \sqrt{\beta_{2x} K} \cos \mu_x & \sqrt{\dfrac{\beta_{2x}}{K}} \sin \mu_x \\ -\sqrt{\dfrac{K}{\beta_{2x}}} (\sin \mu_x + \alpha_{2x} \cos \mu_x) & \sqrt{\dfrac{1}{K\beta_{2x}}} (\cos \mu_x - \alpha_{2x} \sin \mu_x) \end{pmatrix} \tag{6}$$

$$N = \begin{pmatrix} \sqrt{\beta_{2y} K} \cos \mu_y & \sqrt{\frac{\beta_{2y}}{K}} \sin \mu_y \\ -\sqrt{\frac{K}{\beta_{2y}}} \left(\sin \mu_y + \alpha_{2y} \cos \mu_y \right) & \sqrt{\frac{1}{K\beta_{2y}}} \left(\cos \mu_y - \alpha_{2y} \sin \mu_y \right) \end{pmatrix} \tag{7}$$

Substitute (6), (7) into (3), we can derive another form of the flat beam producing condition:

$$\begin{cases} \beta_{2x} = \beta_{2y} \\ \alpha_{2x} = \alpha_{2y} \\ \mu_x = \mu_y - \dfrac{\pi}{2} \end{cases} \tag{8}$$

Section 2. Examples of Flat Beam Producing adaptor

For an ideal cold bunch, once we know the energy of the bunch and the solenoid field, in principle, we can derive the gradient for each quadrupole from the first 3 equations of condition (3). While, since the quadrupole matrix consist of sin, cos, and sinh, cosh terms, these equations are transcendental equations, what I have done is to numerically solve it with my PC. Here are the setup of the adaptor, the results and some analysis of two examples, one is a symmetric-distributed quadrupole sequence, and the other one is somehow modified A0 adaptor, which has different drift length among quadrupoles.

For the symmetric example, the bunch is emitted in the middle of a solenoid, adaptor setup is shown in figure 2 and parameters are shown in table 1. Analytic solutions are in table 2.

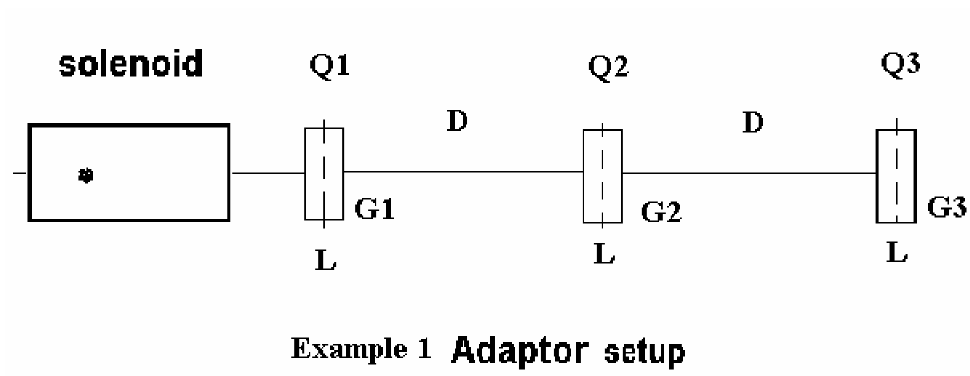

Figure 2. Example 1 Adaptor Setup

Bunch Energy	18 MeV
Solenoid Field	100 Gauss
Quadrupole Length	4 cm
Drift Space	44 cm

Table 1. parameters for example 1

	Gradient 1 (Gauss/cm)	Gradient 2	Gradient 3
Solution 1	231.6	-441.5	2074.6
Solution 2	-231.9	491.5	2067.7

Table 2. solutions for Example 1

Transfer matrix of the adaptor for each solution is:

For solution 1: $$M = \begin{pmatrix} 0.10232 & 0.39699 \\ -1.4297 & 4.2262 \end{pmatrix}, N = \begin{pmatrix} -0.032156 & 1.2632 \\ -0.34231 & -17.651 \end{pmatrix}$$

For solution 2: $$M = \begin{pmatrix} 0.013392 & 2.2339 \\ -0.28268 & 27.518 \end{pmatrix}, N = \begin{pmatrix} -0.18095 & 0.16527 \\ -2.2298 & -3.4899 \end{pmatrix}$$

By using these solution matrices and matrix multiplications, we directly get the x-y space distribution mapping between initial and final status, see figure 3. From figure 3, we found there are two modes of the transformation. More about modes will be disscussed in next section.

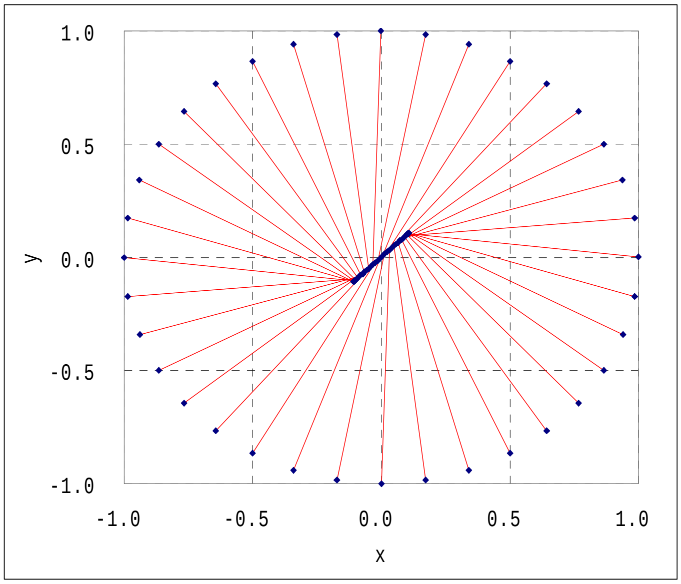

Figure 3.a Initial and final distribution in x-y space by Matrix calculation for solution 1

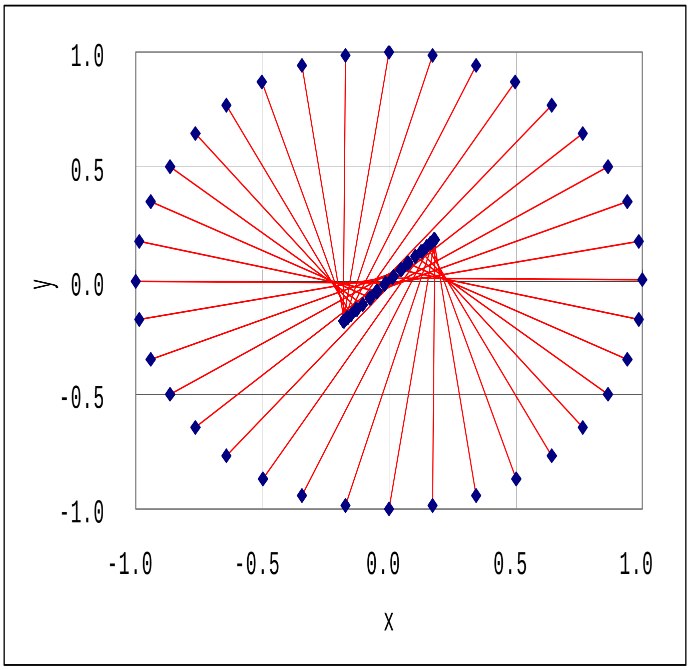

Figure 3.b Initial and final distribution in x-y space by Matrix calculation for solution 2

Beta, alpha functions and phase advances for each solution can be calculated from the transfer matrices of the adaptor, results are shown in figure 4. The phase advance difference between x and y plane is 90 degrees. And alpha, beta functions coincide again when the bunch gets out of the adaptor.

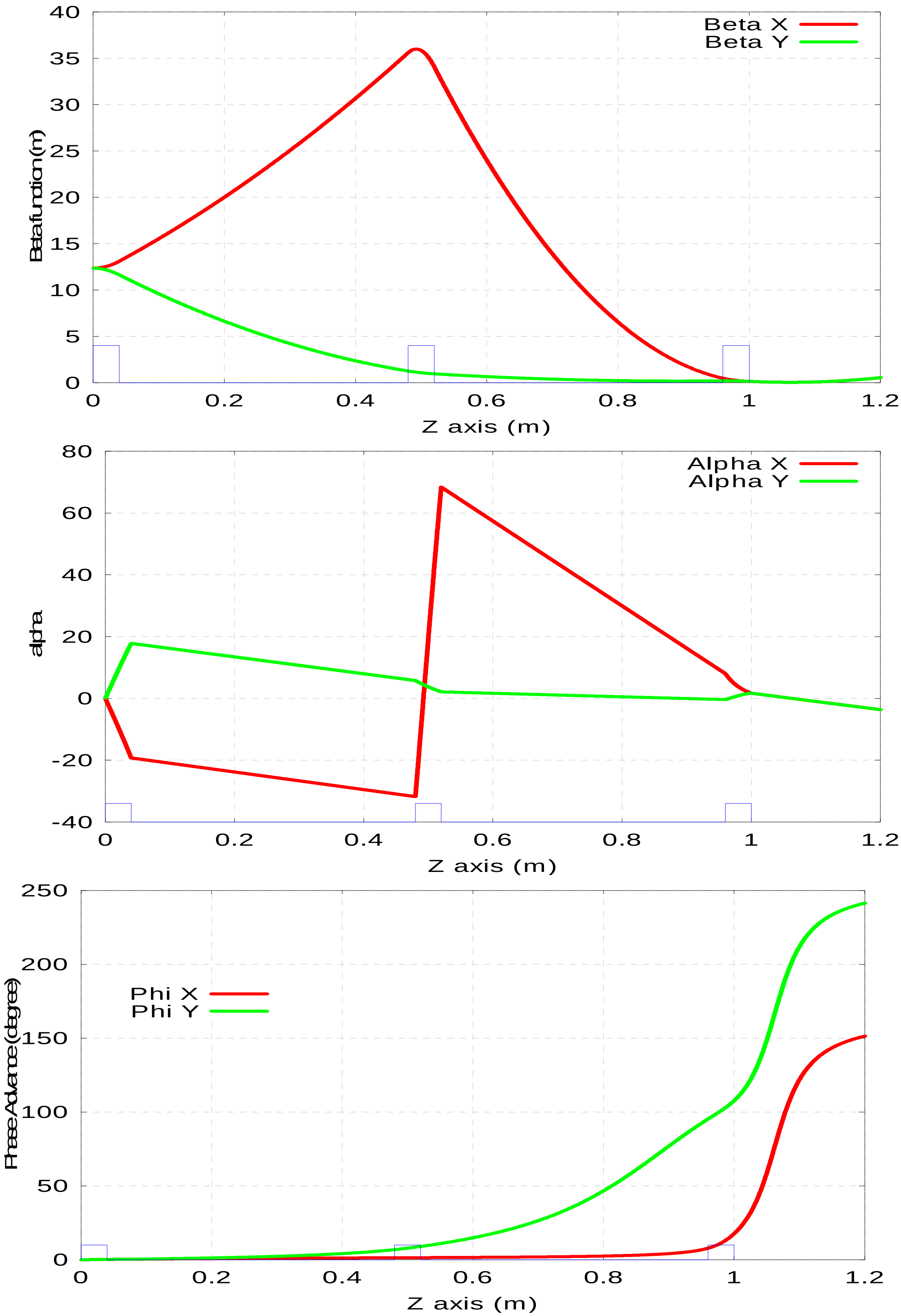

Figure 4.a Beta function and phase advance within the adaptor for solution 1

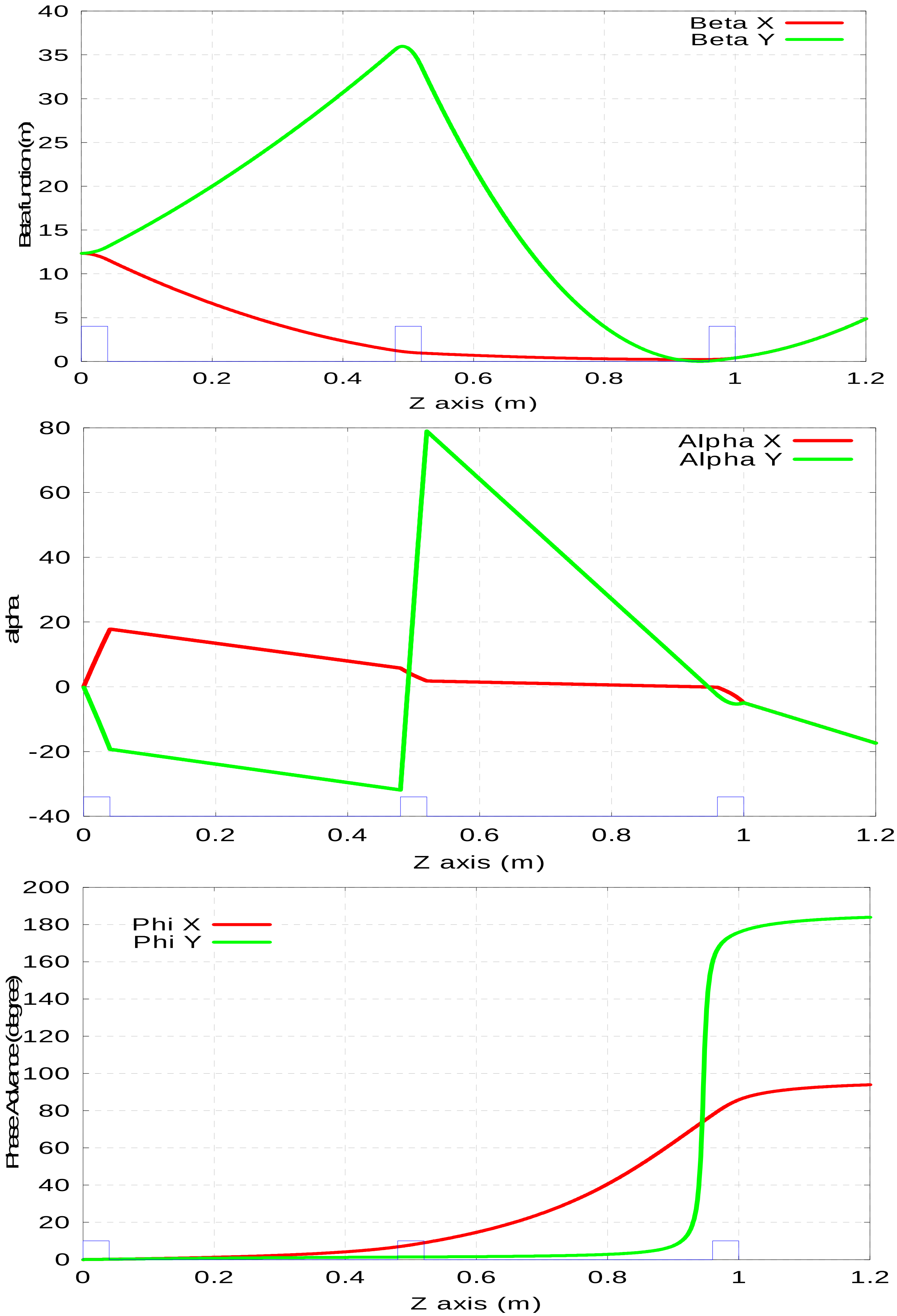

Figure 4.b Beta function and phase advance within the adaptor for solution 2

Second example seems straightforward; all parameters are changed to be same as in A0. While, in A0, the bunch is not directly injected into the adaptor when it get out of the solenoid, instead, the bunch is produced and accelerated in a one-and-half-cell RF-gun, and the solenoid is surrounding the RF-gun, finite longitudinal magnetic field exists where the bunch is born. The solenoid field vanishes before the bunch enters the following 9-cell booster cavity downstream. In RF-gun, the bunch gains 4MeV energy; in the booster cavity, it gains another 12---13MeV energy. Then, the bunch meets the adaptor with about 16.5MeV energy (In my PARMELA simulation, the energy with minimum energy spread is 16.36MeV). Sounds complex compared to last case? We can simplify it.

Notice that the requirement of the adaptor to the bunch is only a matching rotation status when the bunch enters the adaptor and K afore-mentioned is just the angular velocity; we can find the rotation status of the bunch in A0 case by using the conservation of the canonical angular momentum.

$$K = -\varphi' = -\frac{\dot{\varphi}}{v} = -\frac{eBz_{cathode}}{2m_0\gamma\beta c} \qquad (9)$$

So, since at this step I am not going to include any complex factors, such as acceleration, space charge effect…, it turns out to be very simple to investigate the A0 adaptor. I just put a cold bunch of energy 16.36MeV in the middle of a long solenoid which provides a uniform longitudinal magnetic field of the value same as that at cathode in A0 RF-gun.

The modified A0 adaptor setup is shown in figure 5 and parameters are shown in table 3. Analytic solution is in table 4, I found only one set of solution.

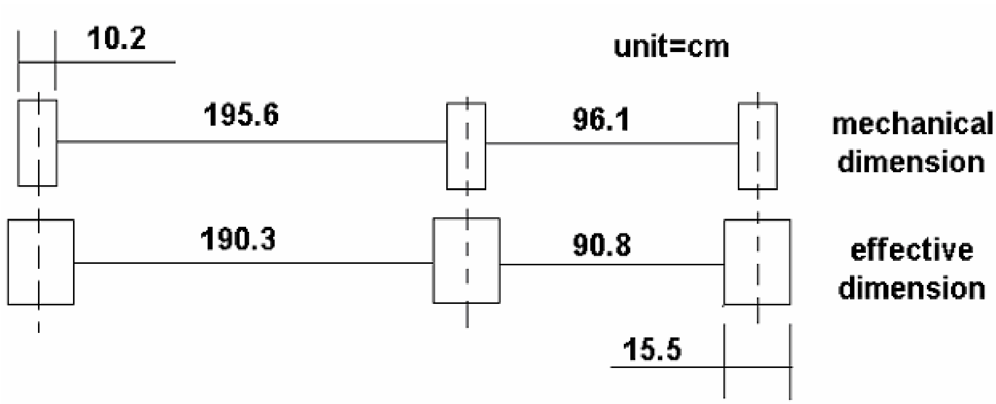

Figure 5. Modified A0 Adaptor Setup

Bunch Energy	16.36MeV
Solenoid Field	783 Gauss
Quadrupole Length(effective)	15.5 cm
Drift Space 1	190.3 cm
Drift Space 2	90.8 cm

Table 3 parameters for example 2

	Gradient 1 (Gauss/cm)	Gradient 2	Gradient 3
Solution 1	30.0	-12.8	8.75

Table 4 solutions for example 2

Transfer matrix of the adaptor for the solution is:

$$M = \begin{pmatrix} 2.6583 & 2.6252 \\ 0.52204 & 0.89173 \end{pmatrix}, N = \begin{pmatrix} -1.8269 & 3.8198 \\ -0.62058 & 0.75014 \end{pmatrix}$$

By using these solution matrices and matrix multiplications, we directly get the x-y space distribution mapping between initial and final status, see figure 6.

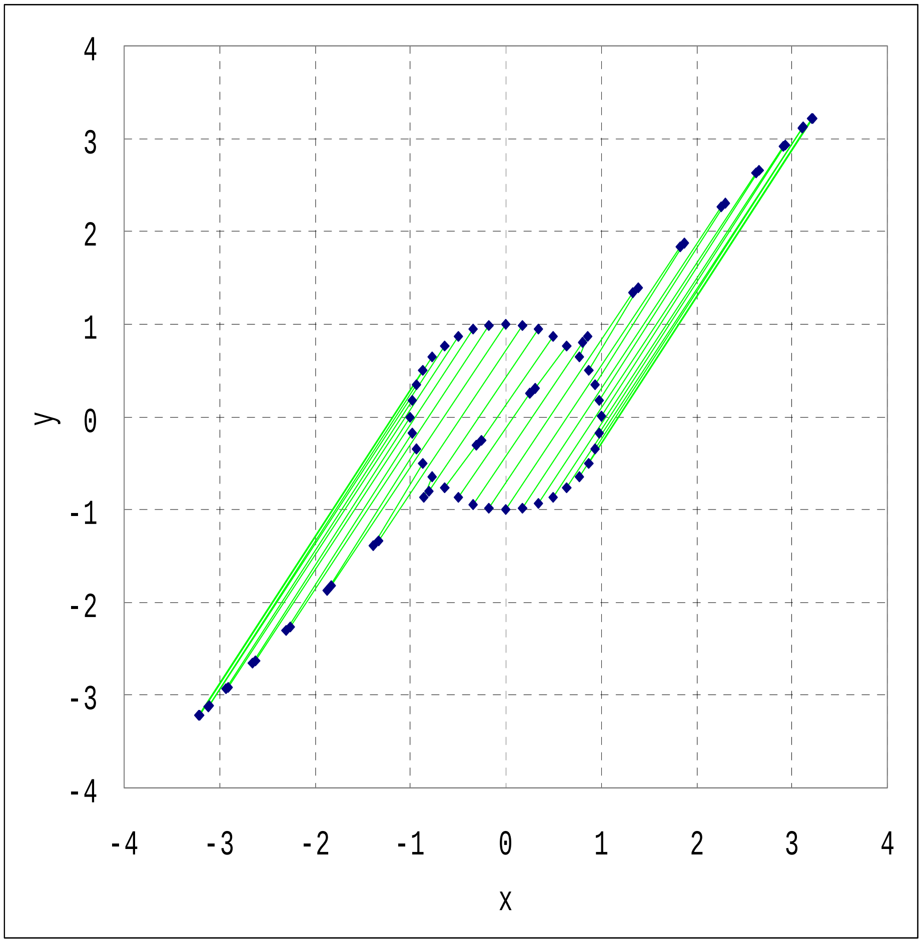

Figure 6 Initial and final distribution in x-y space by Matrix calculation for A0 adaptor

Clearly, the mapping in A0 example is mode 2.

Beta, alpha functions and phase advances for the solution is shown in figure 7.

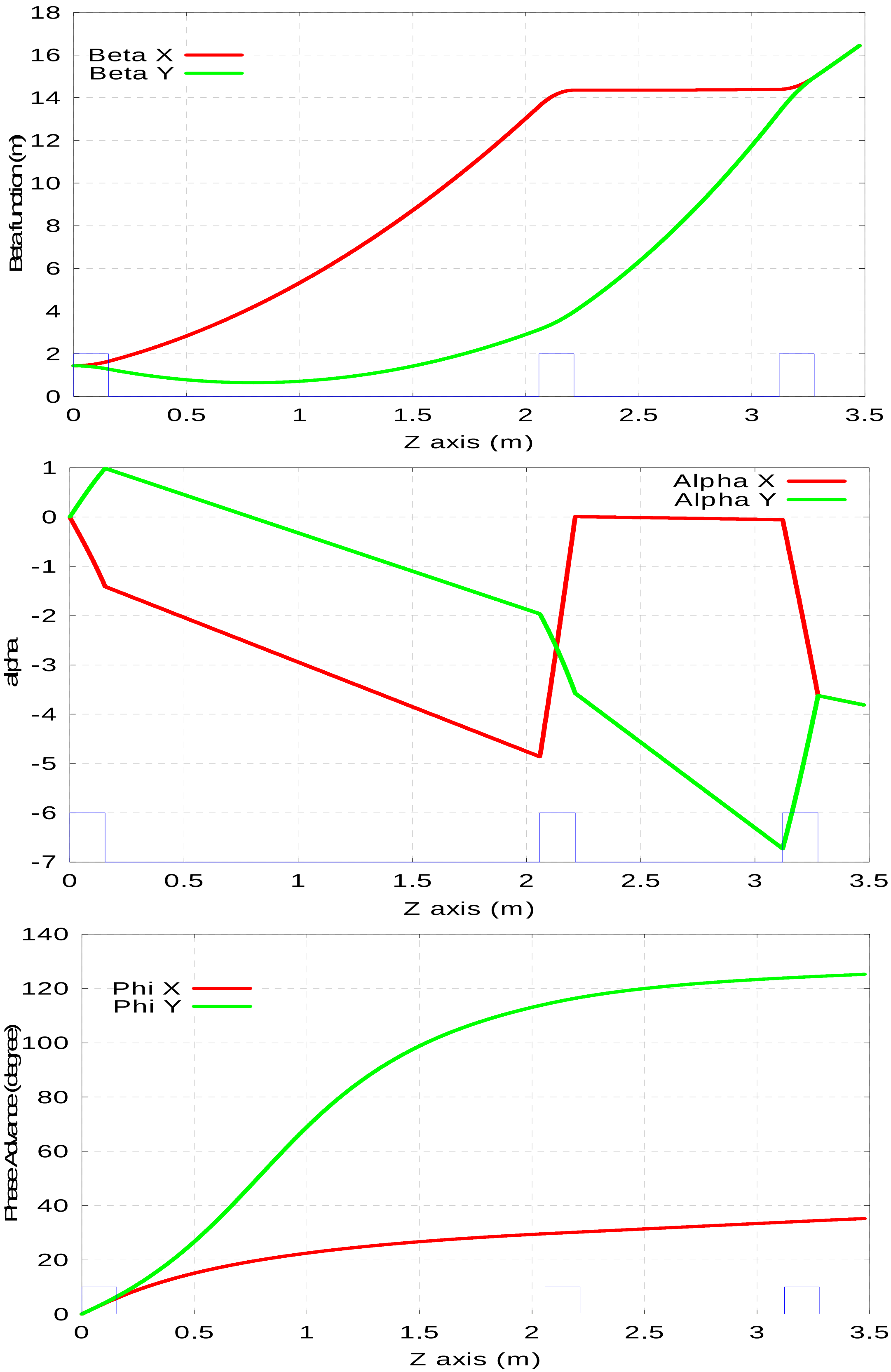

Figure 7 Beta function and phase advance within the A0 adaptor